STRONG CONVERGENCE THEOREM FOR VILENKIN-FEJÉR MEANS

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Abstract. As main result we prove strong convergence theorems of Vilenkin-Fejér means when $0<p\leq 1/2$.

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1. Introduction

It is well-known that Vilenkin system does not form basis in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in L_1-norm. However, in Gát [7] the following strong convergence result was obtained for all $f\in H_1$:

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kf-f\|_1}{k}=0,$$

where S_kf denotes the k-th partial sum of the Vilenkin-Fourier series of f. (For the trigonometric analogue see in Smith [17], for the Walsh-Paley system in Simon [15]). Simon [16] (see also [23]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]}n}\sum_{k=1}^{n}\frac{\|S_kf\|_p^p}{k^{2-p}}\leq c_p\|f\|_{H_p}^p,\qquad (0<p\leq 1) \tag{1}$$

for all $f\in H_p$ and $n\in\mathbb{P}_+$, where $[p]$ denotes integer part of p. In [21] it was proved that sequence $\left\{1/k^{2-p}\right\}_{k=1}^{\infty}$ $(0<p<1)$ in (1) are given exactly.

Weisz [27] considered the norm convergence of Fejér means of Walsh-Fourier series and proved the following:

Theorem W1 (Weisz). Let $p>1/2$ and $f\in H_p$. Then there exists an absolute constant c_p, depending only on p, such that for all $f\in H_p$ and $k=1,2,\dots$

$$\|\sigma_kf\|_p\leq c_p\|f\|_{H_p}.$$

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Theorem W1 implies that

$$\frac{1}{n^{2p-1}}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \qquad (1/2 < p < \infty,\ n = 1, 2, \dots).$$

If Theorem W1 holds for $0 < p \leq 1/2$, then we would have

$$\text{(2)} \qquad \frac{1}{\log^{[1/2+p]} n}\sum_{k=1}^{n}\frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \qquad (0 < p \leq 1/2,\ n = 2, 3, \dots).$$

However, in [18] it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In particular, the following is true:

Theorem T1. There exists a martingale $f \in H_{1/2}$ such that

$$\sup_n \|\sigma_n f\|_{1/2} = +\infty.$$

For the Walsh system in [22] it was proved that (2) holds, though Theorem T1 is not true for $0 < p < 1/2$.

As main result we generalize inequality (2) for bounded Vilenkin systems.

The results for summability of Fejér means of Walsh-Fourier series can be found in [3, 4, 5], [8, 9, 10, 11, 12, 13, 14].

2. Definitions and Notations

Let $\mathbb{P}_+$ denote the set of the positive integers, $\mathbb{P} := \mathbb{P}_+ \cup \{0\}$.

Let $m := (m_{0,} m_1, \dots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

In this paper we discuss bounded Vilenkin groups only, that is

$$\sup_n m_n < \infty.$$

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_k, \dots) \qquad (\ x_k \in Z_{m_k}).$$

It is easy to give a base for the neighbourhood of G_m

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m,\ n \in \mathbb{P})$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $\overline{I_n} := G_m \setminus I_n$.

Let

$$e_n := (0,\dots,0,x_n=1,0,\dots) \in G_m \qquad (n\in\mathbb{P}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0,\dots,0,x_k\neq 0,0,\dots,0,x_l\neq 0,x_{l+1},\dots,x_{N-1},x_N,x_{N+1},\dots), \\ \ k<l<N, \\ I_N(0,\dots,0,x_k\neq 0,0,\dots,0,x_N,x_{N+1},\dots), \\ l=N. \end{cases}$$

and

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2}\bigcup_{l=k+1}^{N-1} I_N^{k,l}\right)\bigcup\left(\bigcup_{k=0}^{N-1} I_N^{k,N}\right). \tag{3}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \qquad (k\in\mathbb{P})$$

then every $n\in\mathbb{P}$ can be uniquely expressed as

$$n=\sum_{j=0}^{\infty} n_j M_j,$$

where $n_j \in Z_{m_j}$ $(j\in\mathbb{P})$ and only a finite number of n_j's differ from zero. Let $|n| := \max\{j\in\mathbb{P};\ n_j\neq 0\}$.

For $n=\sum_{i=1}^{r} s_i M_{n_i}$, where $n_1>n_2>\dots>n_r\geq 0$ and $1\leq s_i<m_{n_i}$ for all $1\leq i\leq r$ we denote

$$\mathbb{A}_{0,2} = \left\{ n\in\mathbb{P}:\ n=M_0+M_2+\sum_{i=1}^{r-2} s_i M_{n_i} \right\}.$$

The norm (or quasi-norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x)\right)^{1/p} \qquad (0<p<\infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}}^p := \sup_{\lambda>0} \lambda^p \mu\{f>\lambda\} < +\infty.$$

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x): G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi\imath x_k/m_k) \qquad \left(\imath^2=-1,\ x\in G_m,\ k\in\mathbb{P}\right).$$

It is known that

$$\sum_{k=0}^{m_n-1} r_n^k(x) = \begin{cases} m_n, & x_n=0, \\ 0, & x_n\neq 0, \end{cases} \tag{4}$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{P})$ on G_m as:

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{P}).$$

Specially, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 24].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\begin{aligned}
\widehat{f}(n) \ &: = \int_{G_m} f\overline{\psi}_n d\mu, \qquad (n \in \mathbb{P}_+) \\
S_n f \ &: = \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k, \quad (n \in \mathbb{P}_+), \\
\sigma_n f \ &: = \frac{1}{n}\sum_{k=1}^{n} S_k f, \qquad (n \in \mathbb{P}_+), \\
D_n \ &: = \sum_{k=0}^{n-1} \psi_k, \qquad (n \in \mathbb{P}_+), \\
K_n \ &: = \frac{1}{n}\sum_{k=1}^{n} D_k, \qquad (n \in \mathbb{P}_+).
\end{aligned}$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{5}$$

and

$$D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{p=m_j-n_j}^{m_j-1} r_j^p. \tag{6}$$

It is well-known that

$$\sup_n \int_{G_m} |K_n(x)|\, d\mu(x) \le c < \infty. \tag{7}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\digamma_n$ $(n \in \mathbb{P})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{P}\right)$ a martingale with respect to $\digamma_n$ $(n \in \mathbb{P})$ (for details see e.g. [25]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{P}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{P})$ is a martingale. Moreover, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{P}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\,\mu(u) \right|$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = \left(f^{(n)}, n \in \mathbb{P}\right)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x)\,\overline{\psi}_i(x)\,d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in \mathbb{P})$ obtained from f.

A bounded measurable function a is p-atom, if there exist a dyadic interval I, such that

$$\int_I a\,d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

3. Formulation of Main Result

Theorem 1. *Let $0 < p \le 1/2$. Then there exists an absolute constant $c_p > 0$, depending only on p, such that for all $f \in H_p$ and $n = 2, 3, \ldots$*

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \le c_p \|f\|_{H_p}^p,$$

where $[x]$ denotes integer part of x.

Corollary 1. *Let $f \in H_{1/2}$. Then*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f - f\|_{1/2}^{1/2}}{k} \to 0, \ \text{as } n \to \infty.$$

Theorem 2. *Let $0 < p < 1/2$ and $\Phi : \mathbb{P}_+ \to [1, \infty)$ is any nondecreasing function, satisfying the conditions $\Phi(n) \uparrow \infty$ and*

$$\overline{\lim_{n \to \infty}} \frac{n^{2-2p}}{\Phi(n)} = \infty. \tag{8}$$

Then there exists a martingale $f \in H_p$, such that

$$\sum_{k=1}^{\infty} \frac{\|\sigma_k f\|_{L_{p,\infty}}^p}{\Phi(k)} = \infty.$$

4. Auxiliary Propositions

Lemma 1. [26] *(see also* [26]*) A martingale* $f=\left(f^{(n)},n\in\mathbb{P}\right)$ *is in* $H_p\,(0<p\leq 1)$ *if and only if there exist a sequence* $(a_k,k\in\mathbb{P})$ *of* p*-atoms and a sequence* $(\mu_k,k\in\mathbb{P})$ *of a real numbers such that for every* $n\in\mathbb{P}$

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f^{(n)}, \tag{9}$$

$$\sum_{k=0}^{\infty}\left|\mu_k\right|^p<\infty.$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}\left|\mu_k\right|^p\right)^{1/p}$, where the infimum is taken over all decomposition of f of the form (9).

Lemma 2. [6] *Let* $n>t,\ t,n\in\mathbb{P},\ x\in I_t\backslash\ I_{t+1}$. *Then*

$$K_{M_n}(x)=\begin{cases}0, & \text{if } x-x_te_t\notin I_n,\\ \frac{M_t}{1-r_t(x)}, & \text{if } x-x_te_t\in I_n.\end{cases}$$

Lemma 3. [19, 20] *Let* $x\in I_N^{k,l},\ k=0,\dots,N-2,\ l=k+1,\dots,N-1$. *Then*

$$\int_{I_N}\left|K_n(x-t)\right|d\mu(t)\leq\frac{cM_lM_k}{nM_N},\quad\text{when } n\geq M_N.$$

Let $x\in I_N^{k,N},\ k=0,\dots,N-1$. Then

$$\int_{I_N}\left|K_n(x-t)\right|d\mu(t)\leq\frac{cM_k}{M_N},\qquad\text{when } n\geq M_N.$$

Lemma 4. *Let* $n=\sum_{i=1}^{r}s_iM_{n_i}$, *where* $n_1>n_2>\cdots>n_r\geq 0$ *and* $1\leq s_i<m_{n_i}$ *for all* $1\leq i\leq r$ *as well as* $n^{(k)}=n-\sum_{i=1}^{k}s_iM_{n_i}$, *where* $0<k\leq r$. *Then*

$$nK_n=\sum_{k=1}^{r}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_j}\right)s_kM_{n_k}K_{s_kM_{n_k}}+\sum_{k=1}^{r-1}\left(\prod_{j=1}^{k-1}r_{n_j}^{s_j}\right)n^{(k)}D_{s_kM_{n_k}}.$$

Proof. It is easy to see that if $k,s,n\in\mathbb{P},\ 0\leq k<M_n$, then

$$D_{k+sM_n}=D_{sM_n}+\sum_{i=sM_n}^{sM_n+k-1}\psi_i=D_{sM_n}+\sum_{i=0}^{k-1}\psi_{i+sM_n}=D_{sM_n}+r_n^sD_k.$$

With help of this fact we get

$$\begin{aligned} nK_n &= \sum_{k=1}^{n} D_k = \sum_{k=1}^{s_1M_{n_1}} D_k + \sum_{k=s_1M_{n_1}+1}^{n} D_k \\ &= s_1M_{n_1}K_{s_1M_{n_1}} + \sum_{k=1}^{n^{(1)}} D_{k+s_1M_{n_1}} \\ &= s_1M_{n_1}K_{s_1M_{n_1}} + \sum_{k=1}^{n^{(1)}} \left(D_{s_1M_{n_1}} + r_{n_1}^{s_1} D_k\right) \\ &= s_1M_{n_1}K_{s_1M_{n_1}} + n^{(1)}D_{s_1M_{n_1}} + r_{n_1}^{s_1} n^{(1)} K_{n^{(1)}}. \end{aligned}$$

If we unfold $n^{(1)}K_{n^{(1)}}$ in similar way, we have

$$n^{(1)}K_{n^{(1)}} = s_2M_{n_2}K_{s_2M_{n_2}} + n^{(2)}D_{s_2M_{n_2}} + r_{n_2}^{s_2} n^{(2)} K_{n^{(2)}},$$

so

$$\begin{aligned} nK_n &= s_1M_{n_1}K_{s_1M_{n_1}} + r_{n_1}^{s_1} s_2M_{n_2}K_{s_2M_{n_2}} + r_{n_1}^{s_1} r_{n_2}^{s_2} n^{(2)} K_{n^{(2)}} \\ &\quad + n^{(1)}D_{s_1M_{n_1}} + r_{n_1}^{s_1} n^{(2)} D_{s_2M_{n_2}}. \end{aligned}$$

Using this method with $n^{(2)}K_{n^{(2)}}, \dots, n^{(r-1)}K_{n^{(r-1)}}$, we obtain

$$\begin{aligned} nK_n &= \sum_{k=1}^{r} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_j}\right) s_kM_{n_k}K_{s_kM_{n_k}} + \left(\prod_{j=1}^{r} r_{n_j}^{s_j}\right) n^{(r)}K_{n^{(r)}} \\ &\quad + \sum_{k=1}^{r-1} \left(\prod_{j=1}^{k-1} r_{n_j}^{s_j}\right) n^{(k)} D_{s_kM_{n_k}}. \end{aligned}$$

According to $n^{(r)} = 0$ it yields the statement of the Lemma 4. □

Lemma 5. [2] *Let* $s, n \in \mathbb{P}$. *Then*

$$D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k.$$

Lemma 6. *Let* $s, t, n \in \mathbb{N},\ n > t,\ s < m_n,\ x \in I_t \backslash I_{t+1}$. *If* $x - x_t e_t \notin I_n$, *then*

$$K_{sM_n}(x) = 0.$$

Proof. In [6] G. Gát proved similar statement to $K_{M_n}(x)=0$. We will use his method. Let $x\in I_t\backslash I_{t+1}$. Using (5) and (6) we have

$$\begin{aligned} sM_nK_{sM_n}(x) &= \sum_{k=1}^{sM_n}D_k(x)=\sum_{k=1}^{sM_n}\psi_k(x)\left(\sum_{j=0}^{t-1}k_jM_j+M_t\sum_{i=m_t-k_t}^{m_t-1}r_t^i(x)\right) \\ &= \sum_{k=1}^{sM_n}\psi_k(x)\sum_{j=0}^{t-1}k_jM_j+\sum_{k=1}^{sM_n}\psi_k(x)M_t\sum_{i=m_t-k_t}^{m_t-1}r_t^i(x) \\ &= J_1+J_2. \end{aligned}$$

Let $k:=\sum_{j=0}^{n}k_jM_j$. Applying (4) we get $\sum_{k_t=0}^{m_t-1}r_t^{k_t}(x)=0$, for $x\in I_t\backslash I_{t+1}$. It follows that

$$J_1=\sum_{k_0=0}^{m_0-1}\cdots\sum_{k_{t-1}=0}^{m_{t-1}-1}\sum_{k_{t+1}=0}^{m_{t+1}-1}\cdots\sum_{k_{n-1}=0}^{m_{n-1}-1}\sum_{k_n=0}^{s-1}\left(\prod_{\substack{l=0\\l\neq t}}^{n}r_l^{k_l}(x)\right)\sum_{j=0}^{t-1}k_jM_j\sum_{k_t=0}^{m_t-1}r_t^{k_t}(x)=0.$$

On the other hand

$$\begin{aligned} J_2 &= \sum_{k_0=0}^{m_0-1}\cdots\sum_{k_{t-1}=0}^{m_{t-1}-1}\sum_{k_{t+1}=0}^{m_{t+1}-1}\cdots\sum_{k_{n-1}=0}^{m_{n-1}-1}\sum_{k_n=0}^{s-1}\left(\prod_{\substack{l=0\\l\neq t}}^{n}r_l^{k_l}(x)\right)M_t\sum_{i=0}^{k_t-1}r_t^i(x) \\ &= \prod_{\substack{l=0\\l\neq t}}^{n-1}\left(\sum_{k_l=0}^{m_l-1}r_l^{k_l}(x)\right)\left(\sum_{k_p=0}^{s}r_p^{k_p}(x)\right)M_t\sum_{i=0}^{k_t-1}r_t^i(x). \end{aligned}$$

Since $x-x_te_t\notin I_n$, at least one of $\sum_{k_l=0}^{m_l-1}r_l^{k_l}(x)$ will be zero, if $l=p\neq t$ and $0\leq p\leq n-1$, that is $J_2=0$. $\square$

5. Proof of the Theorems

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete, if we show that with a constant c_p

$$\frac{1}{\log^{[1/2+p]}n}\sum_{k=1}^{n}\frac{\|\sigma_ka\|_p^p}{k^{2-2p}}\leq c_p<\infty\ (n=2,3,\dots).$$

for every p-atom a, where $[1/2+p]$ denotes the integers part of $1/2+p$. We may assume that a be an arbitrary p-atom with support I, $\mu(I)=M_N^{-1}$ and $I=I_N$. It is easy to see that $\sigma_n(a)=0$, when $n\leq M_N$. Therefore we can suppose that $n>M_N$.

Let $x\in I_N$. Since σ_n is bounded from L_∞ to L_∞ (the boundedness follows from (7)) and $\|a\|_\infty\leq cM_N^{1/p}$ we obtain

$$\int_{I_N}|\sigma_ma(x)|^p\,d\mu(x)\leq c\,\|a\|_\infty^p/M_N\leq c_p<\infty,\quad 0<p\leq 1/2.$$

Hence

$$\frac{1}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{\int_{I_N}|\sigma_m a(x)|^p\,d\mu(x)}{m^{2-2p}} \tag{10}$$
$$\leq \frac{c}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{1}{m^{2-2p}} \leq c_p < \infty.$$

It is easy to show that

$$|\sigma_m a(x)| \leq \int_{I_N}|a(t)|\,|K_m(x-t)|\,d\mu(t)$$
$$\leq \|a\|_\infty \int_{I_N}|K_m(x-t)|\,d\mu(t) \leq cM_N^{1/p}\int_{I_N}|K_m(x-t)|\,d\mu(t).$$

Let $x \in I_N^{k,l}$, $0 \leq k < l < N$. Then from Lemma 3 we get

$$|\sigma_m a(x)| \leq \frac{cM_l M_k M_N^{1/p-1}}{m}. \tag{11}$$

Let $x \in I_N^{k,N}$, $0 \leq k < N$. Then from Lemma 3 we have

$$|\sigma_m a(x)| \leq cM_k M_N^{1/p-1}. \tag{12}$$

Since

$$\sum_{k=0}^{N-2} 1/M_k^{1-2p} \leq N^{[1/2+p]},\ \text{for } 0 < p \leq 1/2$$

by combining (3) and (11-12) we obtain

$$
\begin{aligned}
(13)\qquad & \int_{\overline{I_N}} \left|\sigma_m a(x)\right|^p d\mu(x) \\
= & \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_{j-1}} \int_{I_N^{k,l}} \left|\sigma_m a(x)\right|^p d\mu(x) \\
& + \sum_{k=0}^{N-1}\int_{I_N^{k,N}} \left|\sigma_m a(x)\right|^p d\mu(x) \\
\leq & \; c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1} \frac{m_{l+1}\dots m_{N-1}}{M_N}\frac{\left(M_lM_k\right)^p M_N^{1-p}}{m^p} + \sum_{k=0}^{N-1}\frac{1}{M_N}M_k^pM_N^{1-p} \\
\leq & \; \frac{cM_N^{1-p}}{m^p}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{\left(M_lM_k\right)^p}{M_l} + \sum_{k=0}^{N-1}\frac{M_k^p}{M_N^p} \\
= & \; \frac{cM_N^{1-p}}{m^p}\sum_{k=0}^{N-2}\frac{1}{M_k^{1-2p}}\sum_{l=k+1}^{N-1}\frac{M_k^{1-p}}{M_l^{1-p}} + \sum_{k=0}^{N-1}\frac{M_k^p}{M_N^p} \\
\leq & \; \frac{cM_N^{1-p}N^{[1/2+p]}}{m^p} + c_p.
\end{aligned}
$$

It is easy to show that

$$\sum_{m=M_N+1}^{n}\frac{1}{m^{2-p}} \leq \frac{c}{M_N^{1-p}}, \text{ for } 0<p\leq 1/2.$$

By applying (10) and (13) we get

$$
\begin{aligned}
& \frac{1}{\log^{[1/2+p]} n}\sum_{m=1}^{n}\frac{\left\|\sigma_m a\right\|_p^p}{m^{2-2p}} \\
\leq & \; \frac{1}{\log^{[1/2+p]} n}\sum_{m=M_N+1}^{n}\frac{\int_{\overline{I_N}}\left|\sigma_m a(x)\right|^p d\mu(x)}{m^{2-2p}} \\
& + \frac{1}{\log^{[1/2+p]} n}\sum_{m=M_N+1}^{n}\frac{\int_{I_N}\left|\sigma_m a(x)\right|^p d\mu(x)}{m^{2-2p}} \\
\leq & \; \frac{1}{\log^{[1/2+p]} n}\sum_{m=M_N+1}^{n}\left(\frac{c_pM_N^{1-p}N^{[1/2+p]}}{m^{2-p}} + \frac{c_p}{m^{2-p}}\right) + c_p \\
\leq & \; \frac{c_pM_N^{1-p}N^{[1/2+p]}}{\log^{[1/2+p]} n}\sum_{m=M_N+1}^{n}\frac{1}{m^{2-p}} + \frac{1}{\log^{[1/2+p]} n}\sum_{m=M_N+1}^{n}\frac{1}{m^{2-p}} + c_p \\
\leq & \; c_p<\infty.
\end{aligned}
$$

which completes the proof of Theorem 1.

Proof of Theorem 2. Under condition (8) there exists a sequence of increasing numbers $\{n_k : \ k \geq 0\}$, such that

$$\lim_{k\to\infty} \frac{cn_k^{2-2p}}{\Phi\left(n_k\right)} = \infty.$$

It is evident that for every n_k there exists a positive integer λ_k such that

$$M_{|\lambda_k|+1} \leq n_k < M_{|\lambda_k|+2} \leq \lambda M_{|n_k|+1},$$

where $\lambda = \sup_n m_n$. Since $\Phi\left(n\right)$ is a nondecreasing function we have

$$\overline{\lim_{k\to\infty}} \frac{M_{|\lambda_k|+1}^{2-2p}}{\Phi\left(M_{|\lambda_k|+1}\right)} \geq \lim_{k\to\infty} \frac{cn_k^{2-2p}}{\Phi\left(n_k\right)} = \infty. \tag{14}$$

Applying (14) there exists a sequence $\{\alpha_k : \ k \geq 0\} \subset \{\lambda_k : \ k \geq 0\}$ such that

$$|\alpha_k| \geq 2, \ \text{for} \ \ k \in \mathbb{P}, \tag{15}$$

$$\lim_{k\to\infty} \frac{M_{|\alpha_k|}^{1-p}}{\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)} = \infty \tag{16}$$

and

$$\sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}\left(M_{|\alpha_\eta|+1}\right)}{M_{|\alpha_\eta|}^{1-p}} = m_{|\alpha_\eta|}^{1-p} \sum_{\eta=0}^{\infty} \frac{\Phi^{1/2}\left(M_{|\alpha_\eta|+1}\right)}{M_{|\alpha_\eta|+1}^{1-p}} < c < \infty. \tag{17}$$

Let

$$f_A = \sum_{\{k:\ |\alpha_k|<A\}} \lambda_k a_k,$$

where

$$\lambda_k = \lambda \cdot \frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{M_{|\alpha_k|}^{1/p-1}}$$

and

$$a_k = \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda}\left(D_{M_{|\alpha_k|}+1} - D_{M_{|\alpha_k|}}\right),$$

where $\lambda := \sup_{n\in\mathbb{P}} m_n$. Since

$$S_{M_n} a_k = \begin{cases} a_k, & |\alpha_k| < n, \\ 0, & |\alpha_k| \geq n, \end{cases}$$

and

$$\operatorname{supp}(a_k) = I_{|\alpha_k|}, \qquad \int_{I_{|\alpha_k|}} a_k d\mu = 0, \qquad \|a_k\|_\infty \leq M_{|\alpha_k|}^{1/p} = (\text{supp } a_k)^{-1/p}$$

if we apply Lemma 1 and (17) we conclude that $f \in H_p$.

It is easy to show that

$$\widehat{f}(j) \tag{18}$$

$$= \begin{cases} \Phi^{1/2p}\left(M_{|\alpha_k|+1}\right), & \text{if } j \in \left\{M_{|\alpha_k|}, \dots,\ M_{|\alpha_k|+1}-1\right\},\ k=0,1,2,\dots, \\ 0, & \text{if } j \notin \bigcup\limits_{k=0}^{\infty} \left\{M_{|\alpha_k|}, \dots,\ M_{|\alpha_k|+1}-1\right\}. \end{cases}$$

By using (18) we can write that

$$\sigma_{\alpha_k} f = \frac{1}{\alpha_k} \sum_{j=1}^{M_{|\alpha_k|}} S_j f + \frac{1}{\alpha_k} \sum_{j=M_{|\alpha_k|}+1}^{\alpha_k} S_j f = I + II. \tag{19}$$

It is simple to show that

$$S_j f = \begin{cases} \Phi^{1/2p}\left(M_{|\alpha_0|+1}\right), & \text{if } M_{|\alpha_0|} < j \leq M_{|\alpha_0|+1} \\ 0, & \text{if } 0 \leq j \leq M_{|\alpha_0|}. \end{cases}$$

Suppose that $M_{|\alpha_s|} < j \leq M_{|\alpha_s|+1}$, for some $s = 1,2,...,k$. Then by applying (18) we have that

$$\begin{aligned} S_j f &= \sum_{v=0}^{M_{|\alpha_{s-1}|}} \widehat{f}(v) w_v + \sum_{v=M_{|\alpha_s|}+1}^{j-1} \widehat{f}(v) w_v \\ &= \sum_{\eta=0}^{s-1} \sum_{v=M_{|\alpha_\eta|}}^{M_{|\alpha_\eta|+1}-1} \widehat{f}(v) w_v + \sum_{v=M_{|\alpha_s|}+1}^{j-1} \widehat{f}(v) w_v \\ &= \sum_{\eta=0}^{s-1} \sum_{v=M_{|\alpha_\eta|}}^{M_{|\alpha_\eta|+1}-1} \Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right) w_v + \Phi^{1/2p}\left(M_{|\alpha_s|+1}\right) \sum_{v=M_{|\alpha_s|}+1}^{j-1} w_v \\ &= \sum_{\eta=0}^{s-1} \Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right) \left(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\right) + \Phi^{1/2p}\left(M_{|\alpha_s|+1}\right) \left(D_j - D_{M_{|\alpha_s|}}\right). \end{aligned} \tag{20}$$

Let $M_{|\alpha_s|+1} < j \leq M_{|\alpha_{s+1}|}$, for some $s = 1,2,...,k$. Analogously to (20) we get that

$$S_j f = \sum_{v=0}^{M_{|\alpha_s|+1}} \widehat{f}(v) w_v = \sum_{\eta=0}^{s} \Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right) \left(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}}\right). \tag{21}$$

Let $x \in I_2^{0,1} = (x_0 = 1,\ x_1 = 1,\ x_2, \dots)$. Since (see (5) and Lemma 2)

$$K_{M_n}(x) = D_{M_n}(x) = 0, \quad \text{for} \quad n \geq 2 \tag{22}$$

from (15) and (20)-(21) we obtain that

$$I=\frac{1}{n}\sum_{\eta=0}^{k-1}\Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right)\sum_{v=M_{|\alpha_\eta|}+1}^{M_{|\alpha_\eta|+1}}D_v \tag{23}$$

$$=\frac{1}{n}\sum_{\eta=0}^{k-1}\Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right)\Big(M_{|\alpha_\eta|+1}K_{M_{|\alpha_\eta|+1}}\left(x\right)-M_{|\alpha_\eta|}K_{M_{|\alpha_\eta|}}\left(x\right)\Big)=0.$$

By applying (20), when $s=k$ in II we get that

$$II=\frac{\alpha_k-M_{|n_k|}}{\alpha_k}\sum_{\eta=0}^{k-1}\Phi^{1/2p}\left(M_{|\alpha_\eta|+1}\right)\Big(D_{M_{|\alpha_\eta|+1}}-D_{M_{|\alpha_\eta|}}\Big) \tag{24}$$

$$+\frac{\Phi^{1/2p}\left(M_{|n_k|+1}\right)}{\alpha_k}\sum_{j=M_{|n_k|}+1}^{\alpha_k}\Big(D_j-D_{M_{|n_k|}}\Big)=II_1+II_2.$$

By using (22) we have that

$$II_1=0,\quad \text{for } x\in I_2^{0,1}. \tag{25}$$

Let $\alpha_k\in\mathbb{A}_{0,2}$ and $x\in I_2^{0,1}$. Since $\alpha_k-M_{|\alpha_k|}\in\mathbb{A}_{0,2}$ and

$$D_{j+M_{|\alpha_k|}}=D_{M_{|\alpha_k|}}+w_{M_{|\alpha_k|}}D_j,\ \text{when}\ \ j<M_{|\alpha_k|}$$

By combining (5) Lemmas 4 and 6 we obtain that

$$|II_2|=\frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}\Big(D_{j+M_{|\alpha_k|}}\left(x\right)-D_{M_{|\alpha_k|}}\left(x\right)\Big)\right| \tag{26}$$

$$=\frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}\left|\sum_{j=1}^{\alpha_k-M_{|\alpha_k|}}D_j\left(x\right)\right|$$

$$=\frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}\left|\left(\alpha_k-M_{|\alpha_k|}\right)K_{\alpha_k-M_{|\alpha_k|}}\left(x\right)\right|$$

$$=\frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}\left|M_0K_{M_0}\right|\geq\frac{\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}.$$

Let $0<p<1/2$, $n\in\mathbb{A}_{0,2}$ and $M_{|\alpha_k|}<n<M_{|\alpha_k|+1}$. By combining (19-26) we have that

$$\|\sigma_n f\|_{L_{p,\infty}}^p \geq \frac{c\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)}{\alpha_k^p}\mu\left\{x\in I_2^{0,1}:\ \ |II_2|\geq\frac{c\Phi^{1/2p}\left(M_{|\alpha_k|+1}\right)}{\alpha_k}\right\}$$

$$\geq \frac{c\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)}{\alpha_k^p}\mu\left\{I_2^{0,1}\right\}\geq\frac{c\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)}{M_{|\alpha_k|+1}^p}.$$

By using (16) we get that

$$\sum_{n=1}^{\infty}\frac{\|\sigma_n f\|_{L_{p,\infty}}^p}{\Phi(n)}\geq\sum_{\left\{n\in\mathbb{A}_{0,2}:\ M_{|\alpha_k|}<n<M_{|\alpha_k|+1}\right\}}\frac{\|\sigma_n f\|_{L_{p,\infty}}^p}{\Phi(n)}$$

$$\geq \frac{1}{\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)}\sum_{\left\{n\in\mathbb{A}_{0,2}:\ M_{|\alpha_k|}<n<M_{|\alpha_k|+1}\right\}}\frac{1}{M_{|\alpha_k|+1}^p}$$

$$\geq \frac{cM_{|\alpha_k|}^{1-p}}{\Phi^{1/2}\left(M_{|\alpha_k|+1}\right)}\to\infty,\ \text{when}\quad k\to\infty.$$

Theorem 2 is proved.

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